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A PARAMETER ESTIMATION METHOD FOR THE FLEXURAL WAVE PROPERTIES OF A BEAM

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INTRODUCTION

This paper and corresponding presentation derives an inverse method to measure the complex flexural wavenumber and wave propagation coefficients of a beam. The approach obtains seven measured transfer functions by vibrating the beam transversely with any set of corresponding boundary conditions. These measurements are then combined to yield closed-form solutions of the beam parameters. The test method is subjected to a Monte Carlo simulation, which shows that it is relatively immune to external noise in the data.

SYSTEM MODEL AND INVERSE SOLUTION

The system model of the transverse motion of the beam is the Bernoulli-Euler beam equation, written as

$$EI\frac{\partial^4 u(x,t)}{\partial x^4} + \rho A_b \frac{\partial^2 u(x,t)}{\partial t^2} = 0 \ , \tag{1}$$

where x is the distance along the length of the beam, t is time, u is the displacement of the beam in the (transverse) y-direction, E is the (complex) Young's modulus, I is the moment of inertia, ρ is the density, and A_b is the cross-sectional area of the beam. Implicit in Eqn. (1) is the assumption that plane sections remain plane during bending (or transverse motion). Additionally, Young's modulus, the moment of inertia, the density, and the cross-sectional area remain constant along the entire length of the beam. The displacement is modeled as a steady-state response in time and is expressed as

$$u(x,t) = U(x,\omega)\exp(\mathrm{i}\omega t) \ , \tag{2}$$

where ω is the frequency of excitation (rad/s), $U(x,\omega)$ is the temporal Fourier transform of the transverse displacement, and i is the square root of –1. The temporal solution to Eqn. (1), derived using Eqn. (2) and written in terms of trigonometric functions, is

$$\begin{aligned} U(x,\omega) &= A(\omega)\cos[\alpha(\omega)x] + B(\omega)\sin[\alpha(\omega)x] \\ &\quad + C(\omega)\cosh[\alpha(\omega)x] + D(\omega)\sinh[\alpha(\omega)x], \end{aligned} \tag{3}$$

where $A(\omega)$, $B(\omega)$, $C(\omega)$, and $D(\omega)$ are response coefficients and $\alpha(\omega)$ is the flexural wavenumber given by

$$\alpha(\omega) = \left[\frac{\omega^2}{(EI/\rho A_b)}\right]^{1/4} . \tag{4}$$

For brevity, the ω dependence is omitted from the response coefficients and the flexural wavenumber throughout the remainder of the paper.

Equation (3) has five unknowns and is nonlinear with respect to the unknown flexural wavenumber. It will be shown that the use of seven independent, equally spaced measurements allows the five unknowns to be estimated with closed-form solutions. To begin, all seven frequency-domain transfer functions of acceleration (or displacement) are measured at some location and are then divided by a common reference measurement. Each of the seven transfer functions is collected by two accelerometers placed at different locations on the beam (although one may be placed at the base of the shaker table). The seven measurements are set equal to the theoretical expression given in Eqn. (3). Without loss of generality, the middle measurement location corresponds to $x = 0$ - a location that does not necessarily have to be placed at the middle of the beam. The seven equations are written as

$$T_{-3} = A\cos(3\alpha\delta) - B\sin(3\alpha\delta) + C\cosh(3\alpha\delta) - D\sinh(3\alpha\delta) \tag{5}$$

$$T_{-2} = A\cos(2\alpha\delta) - B\sin(2\alpha\delta) + C\cosh(2\alpha\delta) - D\sinh(2\alpha\delta) \tag{6}$$

$$T_{-1} = A\cos(\alpha\delta) - B\sin(\alpha\delta) + C\cosh(\alpha\delta) - D\sinh(\alpha\delta) \tag{7}$$

$$T_0 = A + C \tag{8}$$

$$T_1 = A\cos(\alpha\delta) + B\sin(\alpha\delta) + C\cosh(\alpha\delta) + D\sinh(\alpha\delta) \tag{9}$$

$$T_2 = A\cos(2\alpha\delta) + B\sin(2\alpha\delta) + C\cosh(2\alpha\delta) + D\sinh(2\alpha\delta) \tag{10}$$

and

$$T_3 = A\cos(3\alpha\delta) + B\sin(3\alpha\delta) + C\cosh(3\alpha\delta) + D\sinh(3\alpha\delta) \tag{11}$$

where δ is the sensor-to-sensor separation distance.

Combining Eqns. (5) through (11) now results in a binomial expression with respect to the cosine function, which is written as

$$a\cos^2(\alpha\delta) + b\cos(\alpha\delta) + c = 0 \ , \tag{12}$$

where

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$$a = 4T_1^2 - 4T_{-1}^2 + 4T_{-2}T_0 - 4T_0T_2 \ , \quad (13)$$

$$b = 2T_{-2}T_{-1} - 2T_{-2}T_1 + 2T_{-1}T_0 - 2T_0T_1 + 2T_{-1}T_2 - 2T_1T_2 + 2T_0T_3 - 2T_{-3}T_0 \ , \quad (14)$$

and

$$c = T_{-1}^2 - T_1^2 + T_2^2 - T_{-2}^2 + T_{-3}T_{-1} - T_{-1}T_3 + T_{-3}T_1 - T_1T_3 + 2T_0T_2 - 2T_{-2}T_0 \ . \quad (15)$$

Equation (12) is now solved using

$$\cos(\alpha\delta) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \phi \ . \quad (16)$$

The inversion of Eqn. (12) allows the complex-valued flexural wavenumber α to be solved as a function of ϕ at every frequency for which a measurement is made. The solution to the real part of α is

$$\mathrm{Re}(\alpha) = \begin{cases} \dfrac{1}{2\delta}\mathrm{Arc}\cos(s) + \dfrac{n\pi}{2\delta} & n\,\text{even} \\ \\ \dfrac{1}{2\delta}\mathrm{Arc}\cos(-s) + \dfrac{n\pi}{2\delta} & n\,\text{odd,} \end{cases} \quad (17)$$

with

$$s = [\mathrm{Re}(\phi)]^2 + [\mathrm{Im}(\phi)]^2 - \sqrt{\{[\mathrm{Re}(\phi)]^2 + [\mathrm{Im}(\phi)]^2\}^2 - \{2[\mathrm{Re}(\phi)]^2 - 2[\mathrm{Im}(\phi)]^2 - 1\}} \quad (18)$$

where n is a non-negative integer, and the capital A denotes the principal value of the inverse cosine function. The value of n is determined from the function s, which is a periodically varying cosine function with respect to frequency. That is, while n is 0 at zero frequency, it increases by 1 every time s cycles through π radians (180º). It is noted here that increasing the integer n allows the estimation process to be used beyond the Nyquist spacing criteria of the sensors because n keeps a record of the number of aliasing cycles between the sensors and thus accounts for these cycles in the measurement process. After the solution to the real part of α is found, the solution to the imaginary part of α is written as

$$\mathrm{Im}(\alpha) = \frac{1}{\delta}\log_e\left\{\frac{\mathrm{Re}(\phi)}{\cos[\mathrm{Re}(\alpha)\delta]} - \frac{\mathrm{Im}(\phi)}{\sin[\mathrm{Re}(\alpha)\delta]}\right\} \ . \quad (19)$$

Although normally considered less important than the estimate of the flexural wavenumber, the response coefficients are next determined. The exact solutions are

$$A = \frac{2T_0\cosh(\alpha\delta) - (T_1 + T_{-1})}{2[\cosh(\alpha\delta) - \cos(\alpha\delta)]} \quad (20)$$

$$B = \frac{2(T_1 - T_{-1})\cosh(\alpha\delta) - (T_2 - T_{-2})}{4\sin(\alpha\delta)[\cosh(\alpha\delta) - \cos(\alpha\delta)]} \quad (21)$$

$$C = \frac{(T_1 + T_{-1}) - 2T_0\cos(\alpha\delta)}{2[\cosh(\alpha\delta) - \cos(\alpha\delta)]} \quad (22)$$

and

$$D = \frac{(T_2 - T_{-2}) - 2(T_1 - T_{-1})\cos(\alpha\delta)}{4\sinh(\alpha\delta)[\cosh(\alpha\delta) - \cos(\alpha\delta)]} \ . \quad (23)$$

NUMERICAL EXAMPLE

The inverse method is examined with a numerical simulation that has noise added to the transfer functions. In this configuration, the beam has both ends constrained to ground with translational springs. The baseline problem has a rectangular cross section with the following physical properties: $E = 10^{11}(1 + 0.05\mathrm{i})$ $\mathrm{N/m^2}$, $\rho = 5000$ $\mathrm{kg/m^3}$, $A_b = 0.015$ $\mathrm{m^2}$, $I = 2.81 \times 10^{-5}$ $\mathrm{m^4}$, $L = 4$ m, $\delta = 0.5$ m, $k_1 = 10^8$ N/m, and $k_2 = 10^{12}$ N/m. Figure 1 is a plot of the estimated and actual values of flexural wavenumber α versus frequency using an error value of e = 0.02. The actual values (no noise) of the real part of α are shown as a solid line, and the estimated values (with noise) of the real part of α are depicted with square symbols. The actual values of the imaginary part of α are shown as a solid line, and the estimated values of the imaginary part of α are depicted with circle symbols.

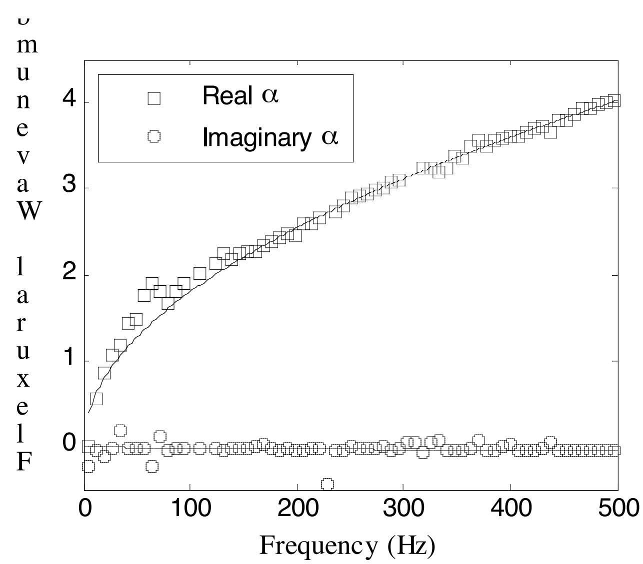

Figure 1. Flexural Wavenumber Versus Frequency

CONSLUSIONS

The parameters of the beam can be estimated with an inverse method that utilizes seven equally spaced measurements. Additionally, an experiment was conducted that verified this method.

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